Model Solutions

1. Let $f(x) = \frac{x^4}{100}$ and $g(x) = x^3$ ... Prove or disprove that $f(x) = O(g(x))$.

   (Dis)Proof by Contradiction. Assume that $f(x) = O(g(x))$.

$$f(x) \in O\big(g(x)\big)$$
$$\exists c, k > 0 \quad \forall x \geq k \quad f(x) \leq cg(x)$$
$$\exists c, k > 0 \quad \forall x \geq k \quad \frac{x^4}{100} \leq cx^3$$
$$\exists c, k > 0 \quad \forall x \geq k \quad \frac{x}{100} \leq c$$

Case #1 $\quad 0 < k < c$

let $x = 101c$

then $x > k$ (since the maximum value of k for this case is less than c)

$$\frac{101c}{100} = 1.01c > c$$
$$\exists x \frac{x}{100} > c$$

Contradiction.

Case #2 $\quad k \geq c$

let $x = 101k$

then $x > k$ (101 times greater, to be precise)

$$\frac{101k}{100} = 1.01k > k > c$$
$$\exists x \frac{x}{100} > c$$

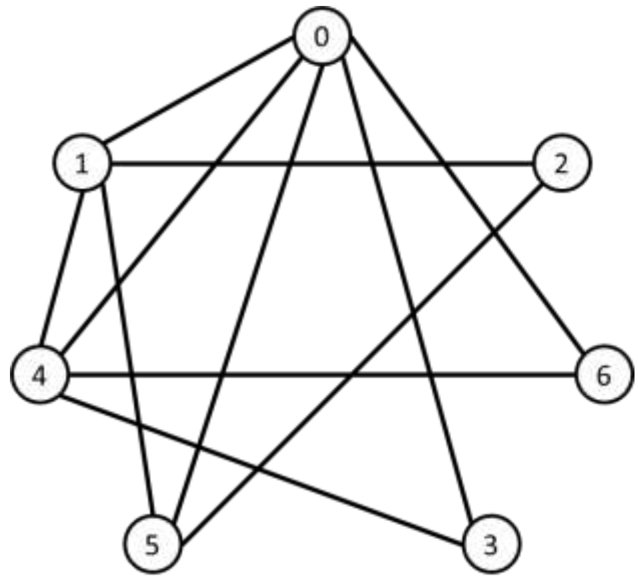
Contradiction.

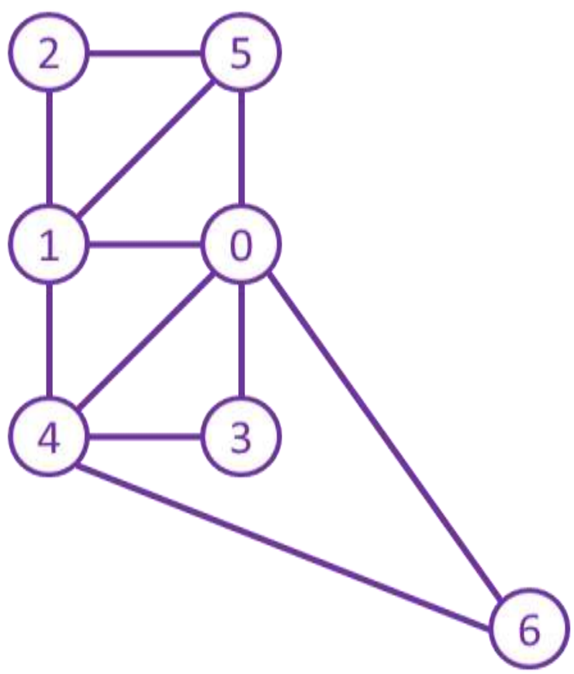
Put less formally, from:

$$\exists c, k > 0 \quad \forall x \geq k \quad \frac{x}{100} \leq c$$

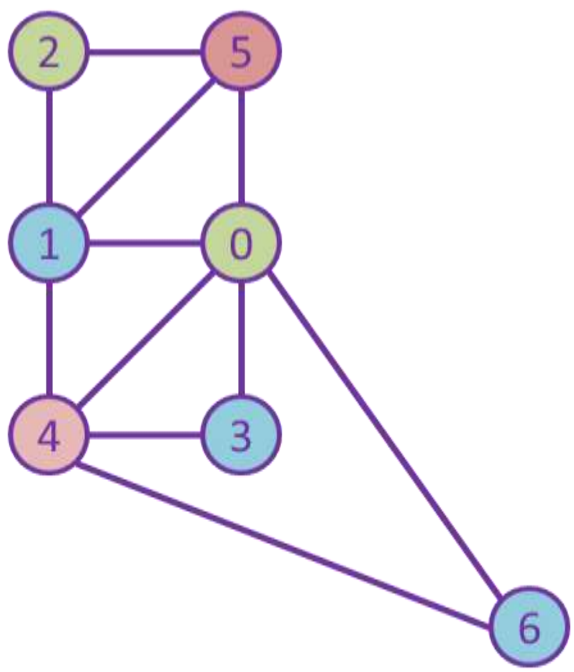
$\frac{x}{100} \leq c$ must hold for all values of $x \geq k$, but there must exist some value of x that is more than 100 times the value of the constant c (regardless of what the constant c actually is). This entails that there must exist at least one value of $x \geq k$ for which the inequality does not hold. This creates the contradiction allowing us to disprove this claim.

2. Create a planar representation for the following graph.





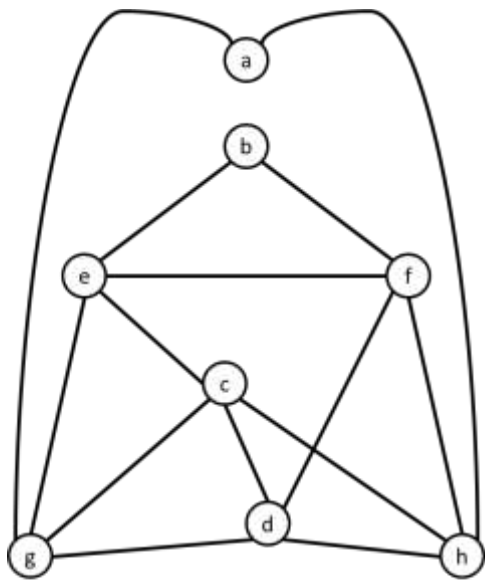
3. Provide a valid colouring for the graph in question 2 that uses the minimum number of colours. (Remember that a valid colouring is one that does not assign the same colour to two adjacent vertices.) Once you have found this colouring, prove that you have used the minimum number of colours. You are not permitted to reference any theorems for answering this questions.



Above is a valid 3-colouring for this graph, as every vertex has been coloured using one of only three distinct colours and no vertex is adjacent to any other vertex with the same colour.

This graph contains subgraph G' = (V', E') where V' = {1, 2, 5} and E' = {{1,2}, {2,5}, {1,5}}. Any valid colouring of the original graph would also need to be valid over this subgraph. Assume that a valid 2-colouring of this graph exists, with two colours named x and y. If vertex 1 is coloured using colour x, then vertex 2 must be coloured using colour y because 1 and 2 are adjacent. If there is a valid 2-colouring then vertex 5 must be coloured using either x or y, but if it is coloured x there is a contradiction because 5 is adjacent to 1 because of the edge {1, 5} and if it is coloured y there is a contradiction because 5 is adjacent to 2 because of the edge {2, 5}. This contradiction indicates that a valid 2-colouring does not exist for G', entailing that the graph cannot be coloured in less than 2 colours either. The minimum number of colours that can be used to colour the original graph cannot be less than the number required for subgraph G', so the minimum number of colours must be 3.

4. An Euler path is a path that uses every edge of a graph exactly once. An Euler cycle is a cycle that uses every edge of a graph exactly once. Does the following graph have an Euler cycle? Does it have an Euler path? If it does, then show it on the graph and if it does not then explain why.



It has several Euler cycles, and since every Euler cycle is, by definition, an Euler path, it has an Euler path as well. If each edge is indicated using the vertex labels for the two vertices connected by that edge, the sequence of edges that forms one of the possible Euler cycles is:

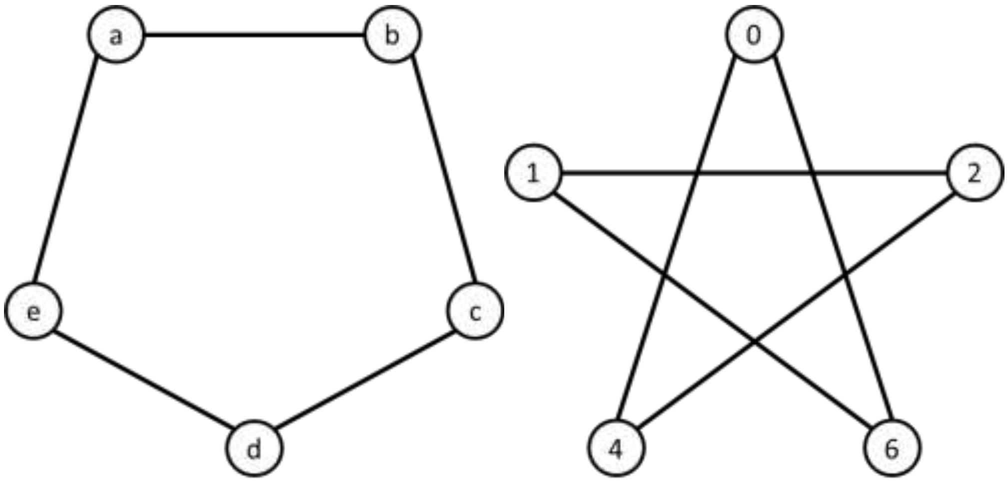
ag
ge
eb
bf
fh
hd
dg
gc
ce
ef
fd
dc
ch
ha

5. A Hamiltonian path is a path that uses every vertex of a graph exactly once. A Hamiltonian cycle is a cycle that uses every edge of a graph exactly once. Does the graph from the preceding question have a Hamiltonian cycle? Does it have a Hamiltonian path? If it does, then show it on the graph and if it does not then explain why.

It has several Hamiltonian cycles as well, and since every Hamiltonian cycle is, by definition, a Hamiltonian path, it has a Hamiltonian path as well. If each edge is indicated using the vertex labels for the two vertices connected by that edge, the sequence of edges that forms one of the possible Euler cycles is:

ag
ge
eb
bf
fd
dc
ch
ha

6. Are the following graphs isomorphic? Prove your answer.



The left graph is (V_L, E_L) where V_L = {a, b, c, d, e} and E_L = {{a,b},{b,c},{c,d},{d,e},{e,a}).

The right graph is (V_R, E_R) where V_R = {0, 1, 2, 4, 6} and E_R = {{0,4},{4,2},{2,1},{1,6},{6,0}).

If the vertices of the right graph were relabelled such that:

Vertex "0" was labelled "a"
Vertex "6" was labelled "b"
Vertex "1" was labelled "c"
Vertex "2" was labelled "d"
Vertex "4" was labelled "e"

Then the right graph would become: $V_{R'}$ = {a, c, d, e, b} and $E_{R'}$ = {{a,e},{e,d},{d,c},{c,b},{b,a}).

Since ($V_{R'}$, $E_{R'}$) = (V_L, E_L), the graphs must be isomorphic.

7. Determine whether or not the following are true and provide a full derivation explaining your answer. The domain of the functions of n below is the positive real numbers. For convenience, you may assume that the logs are in the base of your choice, but you should specify what base you are using in your derivation.

   a. $(7n-2)^2$ is $\Theta(n^2)$

$$if\ n \geq 1$$
$$49n^2 - 28n + 4 < 49n^2 + 28n + 4 \leq 49n^2 + 28n^2 + 4n^2 = 81n^2$$

$$\forall n \geq 1 \quad 49n^2 - 28n + 4 \leq 81n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 49n^2 - 28n + 4 \leq cn^2$$
$$f(x) \in O\big(g(x)\big)$$

$$if\ n \geq 1$$
$$17n^2 = 49n^2 - 28n^2 - 4n^2 \leq 49n^2 - 28n - 4 < 49n^2 - 28n + 4$$
$$\forall n \geq 1 \quad 49n^2 - 28n + 4 \geq 17n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 49n^2 - 28n + 4 \geq cn^2$$
$$f(x) \in \Omega\big(g(x)\big)$$

$$f(x) \in O\big(g(x)\big) \wedge f(x) \in \Omega\big(g(x)\big)$$
$$f(x) \in \Theta\big(g(x)\big)$$

   b. $3n^2 - 8 + n$ is $O(n^2)$

$$if\ n \geq 1$$
$$3n^2 + n - 8 < 3n^2 + n + 8 \leq 3n^2 + n^2 + 8n^2 = 12n^2$$

$$\forall n \geq 1 \quad 3n^2 + n - 8 \leq 12n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 3n^2 + n - 8 \leq cn^2$$
$$f(x) \in O\big(g(x)\big)$$

   c. $\frac{4\log(n+3)}{2}$ is $O(n^2)$

$$if\ n \geq 1$$
$$2\log(n+3) < 2(n+3) < 2n + 6 \leq 2n^2 + 6n^2 = 8n^2$$

$$\forall n \geq 1 \quad 2\log(n+3) \leq 8n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 2\log(n+3) \leq cn^2$$
$$f(x) \in O\big(g(x)\big)$$

d. $1/n - 11 + 8$ is $O(n^2)$

$$if\ n \geq 1$$
$$n^{-1} - 11 + 8 < n^{-1} + 3 \leq n^2 + 3n^2 = 4n^2$$

$$\forall n \geq 1 \quad n^{-1} - 11 + 8 \leq 4n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad n^{-1} - 11 + 8 \leq cn^2$$
$$f(x) \in O\big(g(x)\big)$$

e. $5n^{3/2} - 7n \log n + 8$ is $O(n^3)$

$$if\ n \geq 1$$
$$5n^{3/2} - 7n \log n + 8 < 5n^{3/2} + 7n \log n + 8 \leq 5n^3 + 7n^3 + 8n^3 = 20n^3$$

$$\forall n \geq 1 \quad 5n^{3/2} - 7n \log n + 8 \leq 20n^3$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 5n^{3/2} - 7n \log n + 8 \leq cn^3$$
$$f(x) \in O\big(g(x)\big)$$

f. $2n \log n$ is $O(n^2)$

$$if\ n \geq 1$$
$$2n \log n \leq 2n^2$$

$$\forall n \geq 1 \quad 2n \log n \leq 2n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 2n \log n \leq cn^2$$
$$f(x) \in O\big(g(x)\big)$$

g. $9n^2 + n \log n + 5n^4$ is $\Omega(n^4)$

$$if\ n \geq 1$$
$$5n^4 < 5n^4 + 9n^2 + n \log n$$

$$\forall n \geq 1 \quad 5n^4 + 9n^2 + n \log n \geq 5n^4$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 5n^4 + 9n^2 + n \log n \geq 5n^4$$
$$f(x) \in \Omega\big(g(x)\big)$$

h. $3\log(n^3 + \frac{1}{n^2})$ is $O(n^2)$

$$if\ n \geq 2$$
$$3\log(n^3 + \frac{1}{n^2}) < 3\log(n^3 + 1) < 3\log(n^4) = 12\ \log(n) \leq 12n^2$$

$$\forall n \geq 1 \quad 3\log(n^3 + \frac{1}{n^2}) \leq 12n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 3\log(n^3 + \frac{1}{n^2}) \leq cn^2$$
$$f(x) \in O\big(g(x)\big)$$

i. $9n^2$ is $\Omega(n^2)$

$$if\ n \geq 1$$
$$9n^2 \leq 9n^2$$
$$\forall n \geq 1 \quad 9n^2 \geq 9n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 9n^2 \geq cn^2$$
$$f(x) \in \Omega\big(g(x)\big)$$

j. $8n^2 - 6n$ is $\Omega(1)$

$$if\ n \geq 1$$
$$2n^2 = 8n^2 - 6n^2 \leq 8n^2 - 6n$$
$$\forall n \geq 1 \quad 8n^2 - 6n \geq 2n^2$$
$$\exists c, k > 0 \quad \forall n \geq k \quad 8n^2 - 6n \geq cn^2$$
$$f(x) \in \Omega\big(g(x)\big)$$